Harmonic edge size variation in meshing CAD geometries from IGES format

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Abstract. A mesh generation technique on a closed surface composed of a few parametric surfaces is provided. The edge size function is a fundamental entity in order to be able to apply the process of generalized Delaunay triangulation with respect to the first fundamental form. Unfortunately, the edge size function is not known a-priori in general. We describe an approach which invokes the Laplace-Beltrami operator to determine it. The theory ensuring the functionality of our methods is described. We illustrate our approach by investigating the harmonicity of triangulations of some CAD objects coming directly from IGES files.

Key words: Geometric modeling, IGES, mesh generation, CAD models, Edge size

1 Introduction

We consider the problem of creating a mesh [7] on a surface Γ of a CAD model [1, 6, 8]. In this paper we report on the result of our intensive implementation using IGES (Initial Graphics Exchange Specification) format as CAD exchange. Our main goal focus on the fact that we want the variation in size of the neighboring edges to be smooth. That will result in meshes composed of nicely shaped triangles. The generation of a surface mesh by means of the generalized Delaunay technique [6] with respect to the first fundamental form [2] requires the knowledge of the edge size function which is unfortunately unknown a-priori [10].

Delaunay technique is an efficient method for generating triangulations. In [1], the authors triangulate planar domains using that technique. In [6], a method for evaluating the quality of a mesh is given. Some upper bound of the Delaunay triangulation is given in [4]. In this document, our main contribution is the determination of the edge size function for the Delaunay triangulation. We examine theoretically the functionality of our method. Additionally, our theoretical description is supported by numerical results produced by real IGES data where we investigate mesh harmonicity.

In the next two sections, we will state our problem more specifically and we introduce various important definitions. After quickly giving a motivation for planar problems in section 3, we will detail the meshing of a single parametric surface by using generalization of Delaunay triangulation [4] in section 4. Since the edge size function is known on the boundary, the treatment of the Laplace-Beltrami problem becomes therefore a boundary value problem which we propose to solve numerically in section 5. Section 6 is written for those readers who are interested in theoretical background of the mesh generation approach. At the end of this paper, we will report some benchmarks of CAD objects which are taken from IGES files.

2 Definitions and problem setting

In our approach, the input is a CAD object which is bounded by a closed surface Γ that is composed of n parametric surfaces $\{\mathbf{S}_k\}_{k=1}^n$ such that each $\mathbf{S}_k$ is given as the image of a multiply connected domain $\mathbf{D}_k \subset \mathbf{R}^2$ by the following function

$$\mathbf{x}_k : (u_1, u_2) \in \mathbf{R}^2 \longrightarrow (x_{k,1}(u_1, u_2), x_{k,2}(u_1, u_2), x_{k,3}(u_1, u_2)) \in \mathbf{R}^3 \qquad (1)$$

which is supposed to be bijective and sufficiently smooth [5]. The surfaces $\mathbf{S}_k$ will be referred to as the patches of the whole surface Γ. Every patch of the surface Γ is bounded by a list of curves $\mathbf{C}_i$. The CAD models come from IGES format where the most important nontrivial entities are enlisted in Table 1.

IGES Entities	ID numbers	IGES-codes
Line	110	`LINE`
Circular arc	100	`ARC`
Polynomial/rational B-spline curve	126	`B_SPLINE`
Composite curve	102	`CCURVE`
Surface of revolution	120	`SREV`
Tabulated cylinder	122	`TCYL`
Polynomial/rational B-spline surface	128	`SPLSURF`
Trimmed parametric surface	144	`TRM_SRF`
Transformation matrix	124	`XFORM`

Table 1. Most important IGES entities.

A mesh $\mathbf{M}_h$ is a set of triangles $T_k \subset \mathbf{R}^d$ $(d = 2, 3)$ such that the intersection of two nondisjoint different triangles is either a single node or a complete edge. If d is 2 (resp. 3), then we will call $\mathbf{M}_h$ a 2D (resp. 3D) mesh. For a node A in a mesh $\mathbf{M}_h$, its valence $\eta(A)$ is the number of edges which are incident upon A. The set of nodes which are the endpoints of edges incident upon A, and which are different from A, will be denoted by $\nu(A)$. Our objective is to generate a 3D mesh $\mathbf{M}_h$ such that all nodes of $\mathbf{M}_h$ are located on the surface Γ. Additionally, we aim that the edge lengths varies slowly implying that the lengths of the three edges in any triangle $T \in \mathbf{M}_h$ are proportional.

We will need the matrix $I(\mathbf{x}_k) := [g_{ij}(\mathbf{x}_k)]$ which represents the first fundamental form where

$$g_{ij}(\mathbf{x}_k) :=< \frac{\partial \mathbf{x}_k}{\partial u_i}, \frac{\partial \mathbf{x}_k}{\partial u_j} >= \sum_{p=1}^{3} \frac{\partial x_{k,p}}{\partial u_i} \frac{\partial x_{k,p}}{\partial u_j} \qquad i, j \in \{1, 2\}. \tag{2}$$

3 Motivation for the planar case

In this section, we want to treat briefly the mesh generation problem in the planar case (see Fig. 1) that should provide both motivation and intuitive ideas which facilitate the description of the general case of parametric surfaces. For that matter, we want to triangulate a planar multiply connected domain $\Omega_h \subset \mathbf{R}^2$ with polygonal boundaries P_h. Observe that the boundary edge sizes are generally nonuniform (see Fig. 1). That is usually caused by adaptive discretization of some original curved boundaries P according to some error criteria. In order

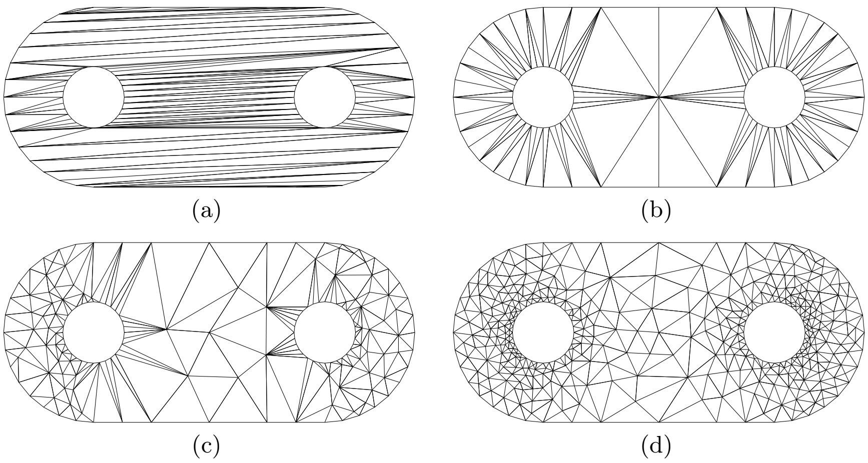


Fig. 1. Selected steps in mesh recursive refinement

to obtain a few of triangles while keeping their good quality shape (Fig. 1(d)), the variation in size of neighboring edges should be small. Let us introduce the edge size function

$$\rho : \Omega_h \longrightarrow \mathbf{R}^+ . \tag{3}$$

If this function is explicitly known, then a way to obtain the mesh is to start from a very coarse mesh (Fig. 1(a)) and to apply Delaunay node insertion [2] in the middle of every edge $[\mathbf{a}, \mathbf{b}]$ whose length exceeds the value of ρ at the midnode of $[\mathbf{a}, \mathbf{b}]$. Unfortunately, the value of ρ is not known in practice. Since the edge size function ρ is known at the boundaries $\partial\Omega_h = P_h$, we consider the following boundary value problem:

$$\Delta\rho := \frac{\partial^2\rho}{\partial u_1^2} + \frac{\partial^2\rho}{\partial u_2^2} = 0 \qquad \text{in} \quad \Omega_h, \tag{4}$$

with the nonhomogeneous Dirichlet boundary condition given by the edge sizes at the boundary. That means the edge size function is required to be harmonic. A harmonic function satisfies in general the mean value property:

$$\rho(a_1, a_2) = \frac{1}{2\pi}\int_0^{2\pi} \rho(a_1 + r\cos\theta, a_2 + r\sin\theta)d\theta. \tag{5}$$

That is, $\rho(\mathbf{a})$ is ideally the same as the average of the values of ρ in a circle centered at $\mathbf{a} = (a_1, a_2)$. Hence, the edge size function ρ has practically small variation.

4 Meshing using the first fundamental form

In this section, we summarize the meshing of a single parametric function $\mathbf{S}_k$ specified by the smooth parametric function $\mathbf{x}_k$ given in (1) To simplify the notation, we will drop the index k in the sequel. The approach in triangulating $\mathbf{S}$ is processed in two steps. First, a 2D mesh on the parameter domain $\mathbf{D}$ is generated according to the first fundamental form. Afterwards, the resulting 2D mesh is lifted to the parametric surface $\mathbf{S}$ by computing its image by $\mathbf{x}$. For that purpose, one starts from a coarse 2D mesh of $\mathbf{D}$ and a generalized two dimensional Delaunay refinement is used as summarized below. We will call an edge of a mesh in the parameter domain a *2D edge* and an edge in the lifted mesh a *3D edge*. Similarly to the planar case, we introduce an edge size function ρ which is defined now on the parametric surface $\rho : \mathbf{S} \longrightarrow \mathbf{R}^+$. By composing ρ with the parameterization $\mathbf{x}$ of $\mathbf{S}$, we have another function $\tilde{\rho} := \rho \circ \mathbf{x}$ which we will call henceforth "parameter edge size function" because it is defined for all $\mathbf{u} = (u, v)$ in the parameter domain. Let us consider a 2D edge $[\mathbf{a}, \mathbf{b}] \subset \mathbf{D}$ and let us denote the first fundamental forms at $\mathbf{a}$ and $\mathbf{b}$ by $I_{\mathbf{a}}$ and $I_{\mathbf{b}}$ respectively. Further, we introduce the following average distance between $\mathbf{a}$ and $\mathbf{b}$

$$d_{\text{Riem}}(\mathbf{a}, \mathbf{b}) := \sqrt{\overrightarrow{\mathbf{ab}}^T T \overrightarrow{\mathbf{ab}}} \qquad T := 0.5(I_{\mathbf{a}} + I_{\mathbf{b}}). \tag{6}$$

The 2D edge $[\mathbf{a}, \mathbf{b}]$ is split if this average distance exceeds the value of the parameter edge size function $\tilde{\rho}$ at the midnode of $[\mathbf{a}, \mathbf{b}]$. Note that no new boundary nodes are introduced during that refinement because only internal edges are allowed to be split. Consider now a 2D edge $[\mathbf{a}, \mathbf{c}]$ is shared by two triangles which form a convex quadrilateral $[a, b, c, d]$. Denote by T the average values of the first fundamental forms $I_{\mathbf{a}}$, $I_{\mathbf{b}}$, $I_{\mathbf{c}}$ and $I_{\mathbf{d}}$ at those nodes. The edge $[\mathbf{a}, \mathbf{c}]$ is flipped into $[\mathbf{b}, \mathbf{d}]$ if the next generalized Delaunay angle criterion is met

$$\|\overrightarrow{\mathbf{bc}} \times \overrightarrow{\mathbf{ba}}\|(\overrightarrow{\mathbf{da}}^T T \overrightarrow{\mathbf{dc}}) < \|\overrightarrow{\mathbf{da}} \times \overrightarrow{\mathbf{dc}}\|(\overrightarrow{\mathbf{cb}}^T T \overrightarrow{\mathbf{ba}}). \tag{7}$$

As in the planar case, ones starts from a very coarse triangulation and one recursively refines or flips edges according to the ideal mesh size function ρ.

We would like now to describe the procedure of splitting $\mathbf{P}$ into a coarse triangulation. Suppose we have a 2D domain $\mathbf{P}$ which may contain some holes and which has polygonal boundaries. First, the initial polygon is split into a few simply connected polygons $\mathbf{P} = \bigcup_{i=1}^{N} \mathbf{P}^{(i)}$. Afterwards, we do the following for every simply connected polygon $\mathbf{P}^{(i)}$. One initializes its set of triangles as empty set $\mathcal{T}_h^{(i)} = \emptyset$. Then, one finds a triangle T which can be chopped off from $\mathbf{P}^{(i)}$. We can repeat that by updating $P^{(i)} := P^{(i)} \setminus T$ and $\mathcal{T}_h^{(i)} = \mathcal{T}_h^{(i)} \cup T$. Finally, the triangulation of $\mathbf{P}$ is the union of all triangulations: $\mathcal{T}_h := \cup_i \mathcal{T}_h^{(i)}$.

5 Edge size function and mesh of CAD surface

Let us consider a parametric surface $\mathbf{S}$ and a differentiable function $F : \mathbf{S} \longrightarrow \mathbf{R}$. The Laplace-Beltrami operator is defined by

$$\Delta_{\mathbf{S}} F = -\frac{1}{\sqrt{g}} \frac{\partial}{\partial u_j} \left(\sqrt{g} g_{ij} \frac{\partial F}{\partial u_i} \right) \tag{8}$$

in which we use Einstein notation in indexing and g is the determinant of I which we introduced in (2). The function F is said to be harmonic if $\Delta_{\mathbf{S}} F = 0$. Since the edge size function ρ should be harmonic, we have the following problem

$$\begin{cases} -\Delta_S \rho = 0 & \text{in } \mathbf{S} \\ \qquad \rho = \rho_{\text{bound}} & \text{on } \partial\mathbf{S}\,. \end{cases} \tag{9}$$

The values of ρ at the boundary which are denoted by ρ_{bound} are known because they are described by the boundary discretization. We will sketch in this section how to numerically solve the boundary value problem (9) by means of the finite element method. For that end, we take a temporary mesh $\mathbf{M}_h$ on $\mathbf{S}$ and we denote its boundary by $\partial\mathbf{M}_h$. For a smooth function ϕ which takes value zero at the boundary we have

$$-\int_{\mathbf{S}} \Delta_{\mathbf{S}} \rho \phi = \int_{\mathbf{S}} < \nabla_{\mathbf{S}} \rho, \nabla_{\mathbf{S}} \phi >=: a(\rho, \phi)\,. \tag{10}$$

Let us define the following set of approximating linear space $V_h := \{f \in \mathbf{C}^0(\mathbf{M}_h) : f_{|T} \in \mathbf{P}_1 \ \forall T \in \mathbf{M}_h\}$, where $\mathbf{C}^0(\mathbf{M}_h)$ denotes the space of functions which are globally continuous on $\mathbf{M}_h$ and $\mathbf{P}_1$ the space of linear polynomials. For a function g we define the set

$$V_h^g := \{f \in V_h : \ f = g \quad \text{on} \quad \partial\mathbf{M}_h\} \tag{11}$$

which is not in general a linear space. The approximated solution ρ_h will reside in the set $V_h^{\rho_{\text{bound}}}$. In order to find ρ_h, we pick an element $\tilde{\rho}$ of $V_h^{\rho_{\text{bound}}}$ and define μ_h by setting $\rho_h = \tilde{\rho} + \mu_h$. The function ρ_h is therefore completely determined if we know the new unknown function μ_h which resides interestingly in V_h^0. Observe that V_h^0 is a linear space in which we choose a basis $\{\phi_i\}_{i\in I}$. As a consequence, the function μ_h is a linear combination of $\{\phi_i\}_{i\in I}$: $\mu_h = \sum_{i\in I} \mu_i \phi_i$. By introducing the following bilinear form $a_h(\cdot,\cdot)$

$$a_h(\psi,\phi) := \sum_{T\in \mathbf{M}_h} a_T(\psi,\phi) \qquad \text{with} \qquad a_T(\psi,\phi) := < \nabla_T \psi, \nabla_T \phi >, \tag{12}$$

we have $a_h(\rho_h,\phi) = 0 \quad \forall \phi \in V_h^0$ or equivalently $a_h(\mu_h,\phi) = -a_h(\tilde{\rho},\phi)$ $\forall \phi \in V_h^0$. Since ϕ_i builds a basis for V_h^0, this leads to a linear equation

$$\sum_{i\in I} a_h(\phi_i,\phi_j)\mu_i = -a_h(\tilde{\rho},\phi_j) \qquad \forall j \in I\,. \tag{13}$$

One can assemble the stiffness matrix $M_{ij} := a_h(\phi_i,\phi_j)$ and solve (13) for μ_i which yields the value of μ_h. For every triangle T in $\mathbf{M}_h$ with internal angles α_1, α_2 and α_3, its contribution to the stiffness matrix M is

$$M_T = 0.5 \begin{bmatrix} \cot\alpha_2 + \cot\alpha_3 & -\cot\alpha_3 & -\cot\alpha_2 \\ -\cot\alpha_3 & \cot\alpha_1 + \cot\alpha_3 & -\cot\alpha_1 \\ -\cot\alpha_2 & -\cot\alpha_1 & \cot\alpha_1 + \cot\alpha_2 \end{bmatrix}. \tag{14}$$

In the previous discussions, the triangulation of a single trimmed surface [3] was provided. Now we describe briefly how to triangulate the surface Γ composed of the surfaces $\mathbf{S}_1,\cdots,\mathbf{S}_n$. First, we discretize the curved boundaries $\mathbf{C}_i$ by piecewise linear curves $\tilde{C}_i$ in which we aim at both accuracy and smoothness: curves which are almost straight need few vertices while those having sharp curvatures need many vertices. Afterwards, we map the 3D nodes of the relevant piecewise linear curves $\tilde{C}_i$ back to the parameter domain $\mathbf{D}_k \subset \mathbf{R}^2$ for each patch $\mathbf{S}_k$. Thus, we may apply the former approach to the 2D preimages of the 3D nodes. In other words, a mesh $\mathbf{M}_k$ for each surface patch $\mathbf{S}_k$ is created by using the technique in section 4. Finally, one merges the meshes $\mathbf{M}_1,\cdots$, $\mathbf{M}_n$ in order to have the final mesh of Γ. Since we use no boundary nodes other than those corresponding to the preimages, no new nodes are inserted during the refinements. As a consequence, nodes at the interface will surely align.

6 Theoretical discussion

We want now to theoretically discuss the applicability of the former triangulation to a given CAD model. Since splitting and flipping an edge are two operations which are admissible once a coarse mesh is given, the main critical points which we want to clarify are twofold. First, we survey the generation of the initial coarse mesh from section 4. Second, we discuss the solvability of the edge size determination in section 5. It has proven that from every simply connected [9] polygon $\mathbf{P}$, one may remove two triangles T_1 and T_2 (called ears) by introducing internal cuts. If we suppose the polygon has n vertices, we need to chop triangles off $(n-2)$ times by applying to have the initial coarse mesh. As a consequence, a simply connected polygon can be triangulated by using only boundary nodes. For triangulation of multiply connected polygons, we need to split them first into several simply connected polygons [9, 10].

Let us now show that the linear system from relation (13) is uniquely solvable. Consider a triangle $T = [A, B, C]$ of the mesh $\mathbf{M}_h$. Let $\boldsymbol{N}_3$ be the unit normal vector of T. Generate two unit vectors $\boldsymbol{N}_1$ and $\boldsymbol{N}_2$ perpendicular to $\boldsymbol{N}_3$ such that $(\boldsymbol{N}_1, \boldsymbol{N}_2, \boldsymbol{N}_3)$ is an orthonormal system which can be centered at A. Since the triangle T is located in the plane spanned by $(\boldsymbol{N}_1, \boldsymbol{N}_2)$, every point $\mathbf{x} \in T$ can be identified by $(v_1, v_2) \in \mathbf{R}^2$ such that $\overrightarrow{A\mathbf{x}} = v_1\boldsymbol{N}_1 + v_2\boldsymbol{N}_2$.

Consider the triangle $t := [(0,0), (1,0), (0,1)]$ and let φ be the parameterization which transforms t into T:

$$\begin{bmatrix} \varphi_1(u_1,u_2) \\ \varphi_2(u_1,u_2) \end{bmatrix} := \begin{bmatrix} V_1 & W_1 \\ V_2 & W_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \tag{15}$$

where (V_1, V_2) and (W_1, W_2) are the components of $W := \overrightarrow{AB}$ and $V := \overrightarrow{AC}$ in $(\boldsymbol{N}_1, \boldsymbol{N}_2)$. Denote by M the above matrix and let θ be the inverse of φ:

$$\begin{bmatrix} \theta_1(v_1,v_2) \\ \theta_2(v_1,v_2) \end{bmatrix} = \frac{1}{\det M} \begin{bmatrix} W_2 & -W_1 \\ -V_2 & V_1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}. \tag{16}$$

Let ψ be the linear polynomial which transforms $\psi(0,0) = \mu(A)$, $\psi(1,0) = \mu(B)$, $\psi(0,1) = \mu(C)$. Its exact expression is $\psi(u_1,u_2) = [\mu(B) - \mu(A)]u_1 + [\mu(C) - \mu(A)]u_2 + \mu(A)$. We want to compute $a_T(\cdot,\cdot)$ in terms of (u_1,u_2). By introducing $a_{ij} := \partial_{v_j}\theta_i$, the integrand for $a_T(\cdot,\cdot)$ involves:

$$\begin{aligned} I(u_1,u_2) &:= (a_{11}\partial_{u_1}\psi + a_{21}\partial_{u_2}\psi)^2 + (a_{12}\partial_{u_1}\psi + a_{22}\partial_{u_2}\psi)^2 && (17)\\ &= (a_{11}^2 + a_{12}^2)(\partial_{u_1}\psi)^2 + 2(a_{11}a_{21} + a_{22}a_{12})(\partial_{u_1}\psi\partial_{u_2}\psi) + && (18)\\ &\quad (a_{21}^2 + a_{22}^2)(\partial_{u_2}\psi)^2. && (19) \end{aligned}$$

Because of equation (16), we have the relations by using $D = \det M$

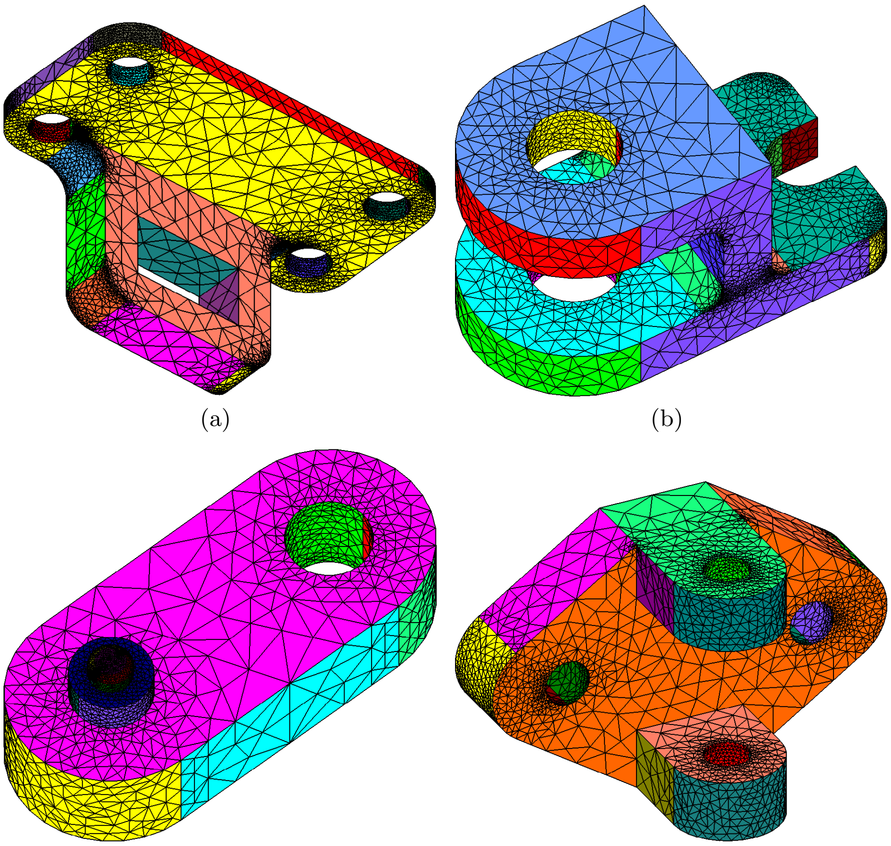
$$a_{11} = W_2/D \qquad a_{12} = -W_1/D \qquad a_{21} = -V_2/D \qquad a_{22} = V_1/D. \tag{20}$$

We have therefore

$$\begin{aligned} I(u_1,u_2) &= \tfrac{1}{D^2}\{(W_2^2 + W_1^2)(\partial_{u_1}\psi)^2 - 2(W_2V_2 + W_1V_1)\partial_{u_1}\psi\partial_{u_2}\psi + \\ &\quad (V_2^2 + V_1^2)(\partial_{u_2}\psi)^2\} \\ &= \tfrac{1}{D^2}\{\|W\|^2(\partial_{u_1}\psi)^2 - 2 < W, V > \partial_{u_1}\psi\partial_{u_2}\psi + \|V\|^2(\partial_{u_2}\psi)^2\} \\ &= \tfrac{1}{D^2}\{[\mu(B) - \mu(A)]^2 < W, W - V > + [\mu(B) - \mu(C)]^2 < V, W > \\ &\quad [\mu(C) - \mu(A)]^2 < V, V - W >\}. \end{aligned}$$

By using the fact that $\cos\alpha = < V, W > /(\|V\| \cdot \|W\|)$ and $\sin\alpha = \det M/(\|V\| \cdot \|W\|)$, we obtain $I(u_1,u_2) = \frac{1}{D}\{[\mu(B) - \mu(C)]^2 \cot\alpha + [\mu(C) - \mu(A)]^2 \cot\beta + [\mu(B) - \mu(A)]^2 \cot\gamma\}$. We have therefore

$$\begin{aligned} a_T(\mu,\mu) &= \int_t I(u_1,u_2)(\det M)\,du_1du_2 && (21)\\ &= 0.5\{[\mu(B) - \mu(C)]^2 \cot\alpha + [\mu(C) - \mu(A)]^2 \cot\beta + && (22)\\ &\quad [\mu(B) - \mu(A)]^2 \cot\gamma\}. && (23) \end{aligned}$$



(a) (b) (c) (d)

Fig. 2. Four meshes generated from CAD objects

By denoting $\Psi(\mu) := a_T(\mu, \mu)$, the system in (14) can be obtained by

$$a_T(\mu, \nu) = 0.5[\Psi(\mu + \nu) - \Psi(\mu) - \Psi(\nu)]. \tag{24}$$

Denote by $\tilde{W} := \frac{1}{D}(\partial_{u_1}\psi)W$ and $\tilde{V} := \frac{1}{D}(\partial_{u_2}\psi)V$, we have $I(u_1, u_2) = \|\tilde{W}\|^2 - 2 < \tilde{W}, \tilde{V} > +\|\tilde{V}\|^2$. By the relation of Cauchy-Schwarz, $I(u_1, u_2) = 0$ iff $\tilde{W} = \lambda\tilde{V}$ for some $\lambda \in \mathbf{R}$. The fact that W and V are not parallel implies that $I(u_1, u_2) = 0$ if $\partial_{u_1}\psi = \partial_{u_2}\psi = 0$. That is to say $\psi = C^{te}$ which means $\mu(A) = \mu(B) = \mu(C)$. Since $\mu \in V_h^0$ is globally continuous and it takes zero values at the boundary as introduced in relation (11), we have $\mu = 0$. The form $a_h(\cdot, \cdot)$ is thus symmetric positive definite. Hence, the linear system from section 5 is solvable and the edge size function can be deduced.

7 Numerical results

In order to investigate the performance of our method, we have implemented a program using C/C++, OpenGL and GLUT. The CAD objects whose surfaces have to be triangulated are given as input in IGES files. We consider four CAD objects which have respectively 30, 25, 24 and 26 patches. We used the former method to generate meshes on their surfaces. The resulting meshes, having respectively 11834, 7944, 7672 and 8932 elements, as portrayed in Fig. 2. We would like to investigate the harmonicity of the meshes which we want to define now. For any considered node $A \in \mathbf{R}^3$ of a mesh $\mathbf{M}_h$, we define

$$\rho(A) := \frac{1}{\eta(A)} \sum_{B\in\nu(A)} \|\overrightarrow{AB}\| \tag{25}$$

to be the average edge length. Now we define $r(A)$ to be the length of the shortest edge incident to a node A and we let s_i be the intersection of the i-th edge incident to A and the sphere centered at A with radius $r(A)$. We define the discrete mean value $\rho_{\mathrm{mean}}(A)$ to be

$$\rho_{\mathrm{mean}}(A) := \frac{1}{\eta(A)} \sum_{B\in\nu(A)} \rho(s_i) \tag{26}$$

in which $\rho(s_i)$ is the following convex combination of $\rho(A)$ and $\rho(B_i)$

$$\rho(s_i) := \frac{\|\overrightarrow{As_i}\|}{\|\overrightarrow{AB}\|}\rho(B_i) + \left(1 + \frac{\|\overrightarrow{As_i}\|}{\|\overrightarrow{AB}\|}\right)\rho(A).$$

	Average harmonicity	Smallest harmonicity	Largest harmonicity
mesh1	0.997292	0.738332	1.263884
mesh2	0.997050	0.724865	1.231922
mesh3	0.997709	0.755730	1.239204
mesh4	0.997353	0.745304	1.270894

Table 2. Harmonicity of the four meshes

We have a discrete mean value property if $\rho(A) = \rho_{\mathrm{mean}}(A)$. We define the harmonicity of a node A to be the ratio $\xi(A) := \rho(A)/\rho_{\mathrm{mean}}(A)$. If the value of the harmonicity approaches the unity, the discrete mean value property is valid. We have computed the average harmonicity of the four meshes and the results can be found in the next table. As one can note in Table 2, the mesh sizes in our tests practically satisfy good harmonicity. That can equally be observed in Fig. 2.

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