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COONS'GENERALIZATION FOR POLYTOPES

INS Preprint

1. Introduction

Over the last decades, several efforts have been done to obtain efficient representations of geometric data with the help of scarce information. That can be illustrated by the representation of complete Bézier and spline [6] curves or surfaces where one stores only a finite number of control points. Similarly, the original transfinite interpolation initiated by Coons can be used to describe a surface by using only the curves which bound it. We want to address in this document the problem of constructing a transfinite interpolation in a convex domain.

Let us first consider a few earlier works related to polyhedra or transfinite interpolations. Polyhedra have already been investigated in the ancient era. The simplest examples might be the Platonic solids which have been examined by Plato. The study of polyhedra and polytopes are usually closely connected to the Euler-Poincaré characteristic [**?**, 13] as well studied by Schläfli [19]. Ziegler has also done a lot of investigation about polytopes and their properties [**?**]. The first who initiated the idea of such transfinite interpolations was Coons [3] in the year sixties. Later, Forrest has intensively used transfinite interpolation in order to generate curved coordinate systems [7]. Additionally, Farin has introduced an improvement [6] of Coons patches related to variational principles. The traditional approach in generating transfinite interpolations [3] takes the sum of the combinations of data on opposite faces which is then subtracted by some mixed quantities. Gordon has expressed [**?**] that in terms of operators and boolean sums such as $\mathcal{P}\oplus\mathcal{Q}=\mathcal{P}+\mathcal{Q}-\mathcal{P}\mathcal{Q}$. The tetrahedral transfinite interpolation of Mansfield [**?**] still used that construction by combining a node with its opposite face. Additionally, the work of Mansfield does not contain any topologic information whatsoever. The major drawback of Mansfield's approach is that it looses its validity when applied to more general frameworks. In fact, for more complicated polyhedra like the ones illustrated in Fig. 1, the notion of opposite faces does not have any meaning except for a few special cases. Perronnet has considered [12] treatments of transfinite interpolation on a few special polyhedra which are not necessarily of tensor-product type such as tetrahedron and pentahedron. He has treated each polyhedron one by one and he did not provide any general way of representing his results. In addition, his formulas are so lengthy that it is difficult to use them for analytical purpose. As for CAD preparation, Brunnett and Randrianarivony have invested a lot to develop and implement a method which is appropriate for surfaces in integral equations [15]. That consists mainly in decomposing trimmed surfaces [1] into four-sided patches [18] and in using Coons patches to generate mappings. Their methods have already been successfully implemented to CAD and molecular surfaces [16, 17]. Harbrecht and Randrianarivony

[11] have used those surface CAD models for applications in Wavelet Boundary Element Method (BEM) solvers. In the framework of multifaceted surfaces, Loop et al. [?] have devised a method of generalizing Bézier on multifaceted polygons. Besides, the usual Bézier curve using Bernstein polynomials [14] has already been generalized [20] by Seidel to multidimensional simplex. In the opinion of the author, the works of Loop and Seidel fit well with the presented method in this document because their approach enables the definition of the curved boundary faces accurately.

The formula which is presented here does not use combinations of data from opposite faces. Instead, it considers the topologic entities which can still be used even for domains which are not in tensor product structures. In order to present that result, we organize this paper as follows. We start in Section 2 by recalling some interesting facts from the usual 2D Coons map on the unit square. We show there another formulation of the Coons map from topologic perspective. That motivates the construction of the transfinite interpolation in the subsequent sections. That will be followed by a precise demonstration of the problem setting for more general convex domains in Section 3 where we find also some introduction of important notations. To prepare the formulation of the transfinite interpolation, we introduce in Section 4 various auxiliary results in form of lemmas. Among others, we will meet there a projection onto the faces of a convex domain and pertaining properties. The formulation and the proof of the main results are found in Section 5 in which we show also the affine stability of the interpolant. Toward the end of this document, we show eventually some illustrative practical results.

2. Coons map revisited

In this section, we consider the usual [3, 10] Coons patch defined on $[0,1]^2$ from topologic perspective. That will serve as a motivation of the subsequent sections. Let us consider four parametric curves

$$\boldsymbol{\alpha},\ \boldsymbol{\beta},\ \boldsymbol{\gamma},\ \boldsymbol{\delta}\ :[0,1]\longrightarrow\mathbb{R}^2 \tag{2.1}$$

which are supposed to fulfill the next compatibility conditions that are illustrated in Fig. 2(a):

$$\boldsymbol{\alpha}(0)=\boldsymbol{\delta}(0),\qquad \boldsymbol{\alpha}(1)=\boldsymbol{\beta}(0),\qquad \boldsymbol{\gamma}(1)=\boldsymbol{\beta}(1),\qquad \boldsymbol{\gamma}(0)=\boldsymbol{\delta}(1). \tag{2.2}$$

The Coons transfinite interpolation consists in generating a parametric surface $\mathbf{x}(u,v)$ defined on the unit square $[0,1]^2$ such that the boundary of the image of $\mathbf{x}$ coincides with the given four curves:

$$\begin{aligned} &\mathbf{x}(u,0)=\boldsymbol{\alpha}(u), &&\mathbf{x}(u,1)=\boldsymbol{\gamma}(u) &&\forall\, u\in[0,1],\\ &\mathbf{x}(0,v)=\boldsymbol{\delta}(v), &&\mathbf{x}(1,v)=\boldsymbol{\beta}(v) &&\forall\, v\in[0,1]\ . \end{aligned} \tag{2.3}$$

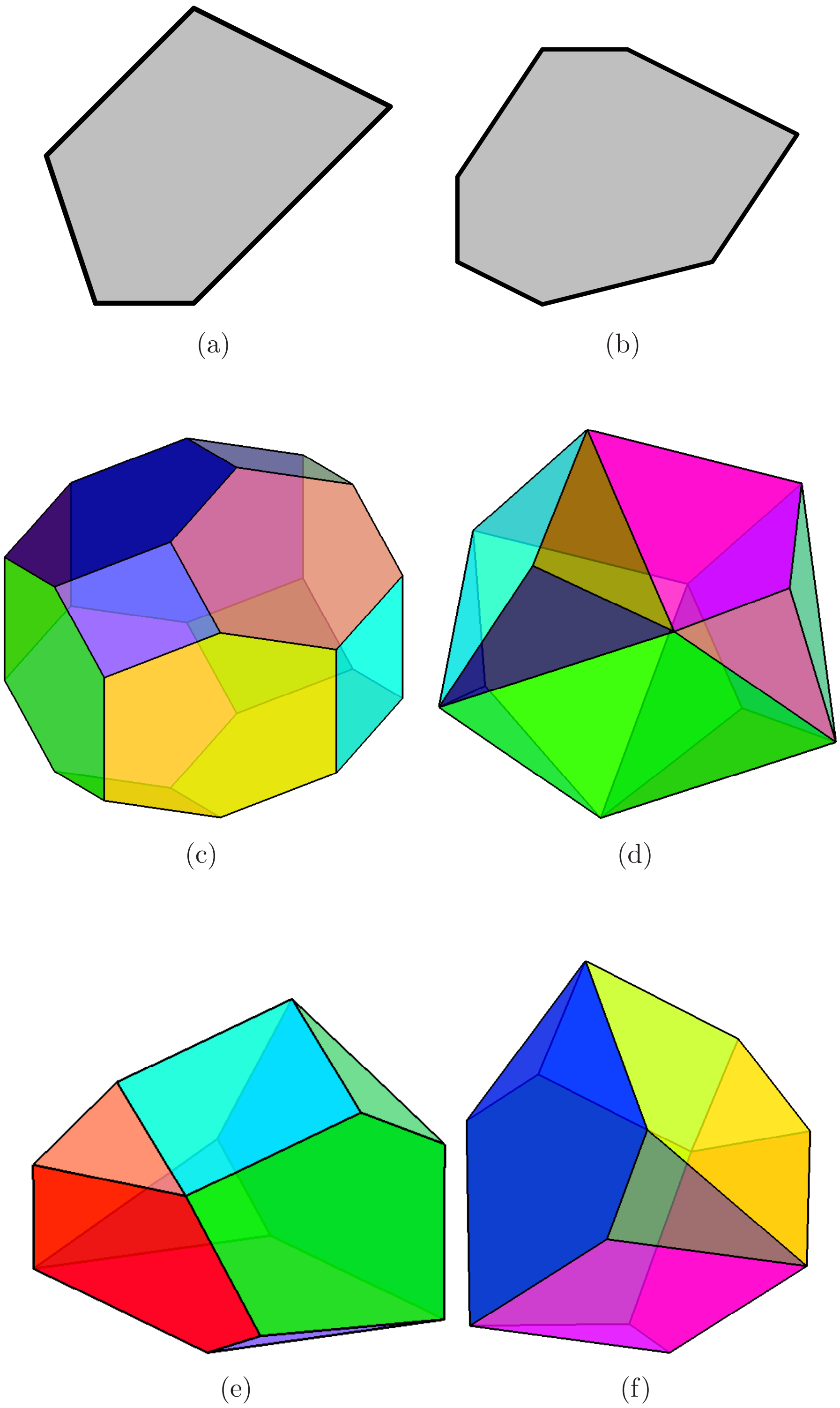
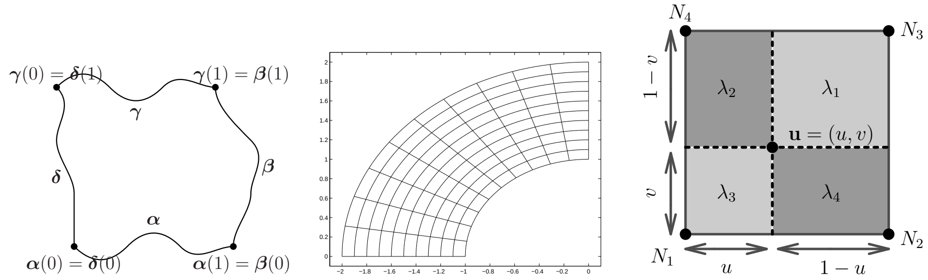


FIGURE 1. A few examples of polytopes. The traditional method of creating a transfinite interpolation by blending data from opposite faces cannot be directly applied to the general case.



(a) (b) (c)

FIGURE 2. (a)Boundary of a Coons patch, (b)Examples of a Coons patch, (c)The four shaded areas determine the barycentric coordinates $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ of $\mathbf{u} = (u, v)$ with respect to $[N_1, N_2, N_3, N_4]$

As an illustration, we can see in Fig. 2(b) the image of a uniform grid on the unit square by a Coons map $\mathbf{x}$.

Let f_0 and f_1 denote two arbitrary smooth functions satisfying

$$f_i(j) = \delta_{ij} \quad i, j = 0, 1 \quad \text{and} \qquad f_0(t) + f_1(t) = 1 \qquad \forall\, t \in [0, 1]. \tag{2.4}$$

The functions f_0, f_1 which are better known as *blending functions* can be chosen in several ways [7]. Among others, three methods are usual: linear, cubic and trigonometric blending functions:

$$\begin{array}{lll} f_0(t) := 1 - t & \text{and} & f_1(t) := t, \\ f_0(t) := B_0^3(t) + B_1^3(t) & \text{and} & f_1(t) := B_2^3(t) + B_3^3(t), \\ f_0(t) := \cos^2(0.5\pi t) & \text{and} & f_1(t) := \sin^2(0.5\pi t). \end{array} \tag{2.5}$$

A construction of a solution to (2.3) which was due to Coons first blends the opposite curves $\boldsymbol{\beta}$ and $\boldsymbol{\delta}$. Then, it is added by a blend between the opposite curves $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$. Finally, those two blends are subtracted by mixed terms to ensure good interpolation at the corners. That can be expressed in matrix form as:

$$\begin{aligned} \mathbf{x}(u, v) \;\; := \;\; & \begin{bmatrix} f_0(u) & f_1(u) \end{bmatrix} \begin{bmatrix} \boldsymbol{\delta}(v) \\ \boldsymbol{\beta}(v) \end{bmatrix} + \begin{bmatrix} \boldsymbol{\alpha}(u) & \boldsymbol{\gamma}(u) \end{bmatrix} \begin{bmatrix} f_0(v) \\ f_1(v) \end{bmatrix} - \\ & \begin{bmatrix} f_0(u) & f_1(u) \end{bmatrix} \begin{bmatrix} \boldsymbol{\alpha}(0) & \boldsymbol{\gamma}(0) \\ \boldsymbol{\alpha}(1) & \boldsymbol{\gamma}(1) \end{bmatrix} \begin{bmatrix} f_0(v) \\ f_1(v) \end{bmatrix}. \end{aligned} \tag{2.6}$$

The above construction is difficult to generalize to more general polytopes except to some special cases such as a simplex. That is because the notion of opposite faces is not obvious to figure out. Now, we want to consider the above transfinite interpolation on the unit square but from topologic point of view. To that end, consider a function

$\mathcal{B}$ defined on the boundary of $\overline{\mathcal{D}} := [0,1]^2$ as follows

$$\mathcal{B}(u,0) := \boldsymbol{\alpha}(u), \quad \mathcal{B}(u,1) := \boldsymbol{\gamma}(u), \quad \mathcal{B}(0,v) := \boldsymbol{\delta}(v), \quad \mathcal{B}(1,v) := \boldsymbol{\beta}(v). \tag{2.7}$$

For a point $\mathbf{u} = (u,v)$ inside $\overline{\mathcal{D}}$, the expressions $\lambda_1 := (1-u)(1-v)$, $\lambda_2 := u(1-v)$, $\lambda_3 := uv$, $\lambda_4 := (1-u)v$ determine barycentric coordinates with respect to $N_1 := (0,0)$, $N_2 := (1,0)$, $N_3 := (1,1)$, $N_4 := (0,1)$ as shown in Fig.2(c). Let us now consider the four functions

$$\begin{aligned} b_1(u,v) &:= f_0(u)f_0(v), \qquad & b_2(u,v) &:= f_1(u)f_0(v), \\ b_3(u,v) &:= f_1(u)f_1(v), \qquad & b_4(u,v) &:= f_0(u)f_1(v). \end{aligned} \tag{2.8}$$

We deduce from (2.4) that they sum to unity and that $b_i(u,v) = 0$ when the barycentric coordinate $\lambda_i = 0$. By using the properties of the blending functions, the above expression of $\mathbf{x}$ can be reformulated in terms of b_i as follows

$$\begin{aligned} \mathbf{x}(u,v) \;&=\; b_1(u,v)\big[\boldsymbol{\alpha}(u) + \boldsymbol{\delta}(v) - \boldsymbol{\alpha}(0)\big] \\ &+\; b_2(u,v)\big[\boldsymbol{\alpha}(u) + \boldsymbol{\beta}(v) - \boldsymbol{\alpha}(1)\big] \\ &+\; b_3(u,v)\big[\boldsymbol{\beta}(v) + \boldsymbol{\gamma}(u) - \boldsymbol{\gamma}(1)\big] \\ &+\; b_4(u,v)\big[\boldsymbol{\gamma}(u) + \boldsymbol{\delta}(v) - \boldsymbol{\gamma}(0)\big], \end{aligned} \tag{2.9}$$

or equivalently

$$\begin{aligned} \mathbf{x}(u,v) \;&=\; (-1)^3\Big\{b_1(u,v)\big[(-1)^1\mathcal{B}(u,0) + (-1)^1\mathcal{B}(0,v) + (-1)^0\mathcal{B}(0,0)\big] \\ &+\; b_2(u,v)\big[(-1)^1\mathcal{B}(u,0) + (-1)^1\mathcal{B}(1,v) + (-1)^0\mathcal{B}(1,0)\big] \\ &+\; b_3(u,v)\big[(-1)^1\mathcal{B}(1,v) + (-1)^1\mathcal{B}(u,1) + (-1)^0\mathcal{B}(1,1)\big] \\ &+\; b_4(u,v)\big[(-1)^1\mathcal{B}(u,1) + (-1)^1\mathcal{B}(0,v) + (-1)^0\mathcal{B}(0,1)\big]\Big\}. \end{aligned}$$

The role of this document is to generalize the above formula to general convex domains in $\mathbb{R}^d$ having M nodes N_1,...,N_M. In fact, we will develop a transfinite interpolation of the form

$$(-1)^{d+1}\sum_{i=1}^{M} b_i(\boldsymbol{\lambda}) \sum_{\Pi\in\mathcal{G}(i)} (-1)^{\dim(\Pi)}\mathcal{B}\circ\mathbb{P}_{\Pi,N_i}(\boldsymbol{\lambda}) \tag{2.10}$$

where $\mathcal{G}(i)$ is the set of topologic entities (nodes and edges in 2D) such that the node N_i is incident upon them and $\mathbb{P}$ is a certain projection which will be specified subsequently.

3. Nomenclatures and problem setting

There are several ways of defining a polytope but the two most applied settings are the one using convex hulls and the one using hyperplanes. They correspond to the

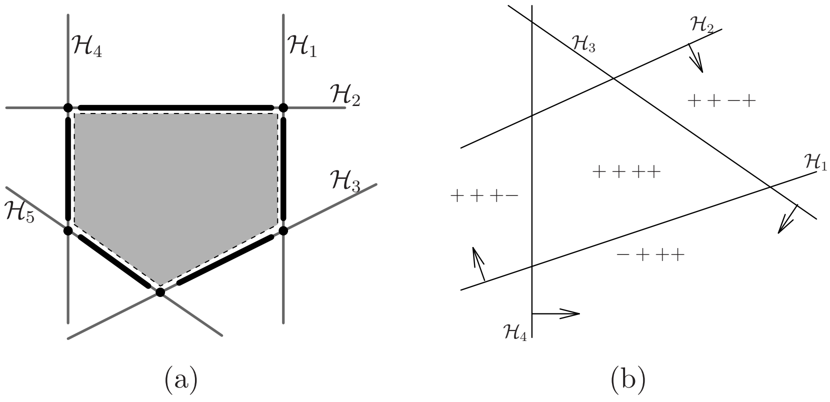


FIGURE 3. (a)The large points, the bold line segments and the shaded domain represent some faces $\bigcap_{i=1}^{4} \mathcal{H}_i^{\sigma_i}$ where $\sigma_i \in \{0, +, -\}$, (b)Some chambers identified by their sign sequence $\sigma_1\sigma_2\sigma_3\sigma_4$ split $\mathbb{R}^d \setminus \{\cup_{i=1}^{4} \mathcal{H}_i\}$.

so-called $\mathcal{V}$-polytope and $\mathcal{H}$-polytope. Thus, a polytope $\overline{\mathcal{D}}$ can be expressed in the following forms

$$(\mathcal{V} - \text{polytope}) \quad : \quad \overline{\mathcal{D}} = \text{HULL}\big\{N_i \in \mathbb{R}^d : \; i = 1, ..., M\big\}, \tag{3.11}$$

$$(\mathcal{H} - \text{polytope}) \quad : \quad \overline{\mathcal{D}} = \overline{\bigcap_{i \in \mathcal{L}} \mathcal{H}_i^+}. \tag{3.12}$$

In both cases, we assume that polytopes are closed and bounded. The famous theorem [?, ?, ?] of *Weyl-Minkowski* states the equivalence of those two formulations. Consequently, we will use both settings and choose at each circumstance the one which facilitates the presentation. In order that this paper is readable by nonspecialits in polytopes, let us specify those two settings more accurately by introducing important notations and properties.

3.1. **Hyperplane arrangements and polytopes.** Let us consider an *arrangement* $\mathcal{A}$ which [?] is defined to be a finite set of (affine) hyperplanes $\mathcal{H}_i$ in $\mathbb{R}^d$ where $i \in \mathcal{L}$. Each hyperplane $\mathcal{H}_i$ splits $\mathbb{R}^d$ into two half-spaces $\mathcal{H}_i^+$ and $\mathcal{H}_i^-$ which are both supposed to be topologically open. Thus, the hyperplane $\mathcal{H}_i$ belongs to neither $\mathcal{H}_i^+$ nor $\mathcal{H}_i^-$. In order to manage the sign distributions, let us denote $\mathcal{H}_i^0 := \mathcal{H}_i$. More precisely, there are some $a_1^i, \cdots, a_d^i, a_{d+1}^i \in \mathbb{R}$ such that

$$\mathcal{H}_i^0 \;:=\; \{\mathbf{u} = (u_1, ..., u_d) \in \mathbb{R}^d : \quad a_1^i u_1 + \cdots + a_d^i u_d + a_{d+1}^i = 0\}, \tag{3.13}$$

$$\mathcal{H}_i^+ \;:=\; \{\mathbf{u} = (u_1, ..., u_d) \in \mathbb{R}^d : \quad a_1^i u_1 + \cdots + a_d^i u_d + a_{d+1}^i > 0\}, \tag{3.14}$$

$$\mathcal{H}_i^- \;:=\; \{\mathbf{u} = (u_1, ..., u_d) \in \mathbb{R}^d : \quad a_1^i u_1 + \cdots + a_d^i u_d + a_{d+1}^i < 0\}. \tag{3.15}$$

The hyperplanes $\mathcal{H}_i$, $i \in \mathcal{J}$ decompose the whole space $\mathbb{R}^d$ into nonempty subsets F of the form

$$F = \bigcap_{i\in\mathcal{L}} \mathcal{H}_i^{\sigma_i}, \qquad \sigma_i \in \{0, +, -\} \tag{3.16}$$

which are called *faces* as illustrated in Fig. 3(a). A chamber (see Fig. 3(b)) is a face for which $\sigma_i \neq 0$ for all $i \in \mathcal{L}$. Note that chambers are open and convex and they form a partition of $\mathbb{R}^d \setminus (\cup_{i\in\mathcal{L}}\mathcal{H}_i)$ as illustrated in Fig. 3(b). A *polytope* $\overline{\mathcal{D}}$ is defined to be the closure of a *bounded* chamber:

$$\mathcal{D} = \bigcap_{i\in\mathcal{L}} \mathcal{H}_i^{\sigma_i} \qquad \sigma_i \in \{+, -\}. \tag{3.17}$$

The faces of $\mathcal{D}$ having dimension $d-1$ will be termed *facets* while those having zero dimension simply *nodes* or *vertices*. For three index subsets $I_1, I_2, I_3 \subset \mathcal{L}$ where $I_i \cap I_j = \emptyset$, one defines the *dimension* of a subset of the form

$$B = \left[\cap_{i\in I_0} \mathcal{H}_i^0\right] \bigcap \left[\cap_{i\in I_1} \mathcal{H}_i^+\right] \bigcap \left[\cap_{i\in I_2} \mathcal{H}_i^-\right] \subset \mathbb{R}^d \tag{3.18}$$

to be the dimension of $\cap_{i\in I_0}\mathcal{H}_i^0$ as illustrated in Fig. 4(a). If $I_0 = \emptyset$, then we suppose $\cap_{i\in I_0}\mathcal{H}_i^0 := \mathbb{R}^d$ so that $\dim(B) = d$. In particular, as seen from (3.17) the dimension of a polytope $\mathcal{D}$ in $\mathbb{R}^d$ is d. Let us now consider the incidence in the set $\mathcal{F}$ of faces of an arrangement $\mathcal{A}$. An element $\Pi_1 \in \mathcal{F}$ is called a face of $\Pi_2 \in \mathcal{F}$ if for each $i \in \mathcal{L}$, one has (see Fig. 4(b))

$$\sigma_i(\Pi_1) = 0 \qquad \text{or} \qquad \sigma_i(\Pi_1) = \sigma_i(\Pi_2). \tag{3.19}$$

In such a case, we denote $\Pi_1 \sqsubseteq \Pi_2$ and we say Π_1 is *incident* upon Π_2. The relation between a face and the faces incident upon it is [**?**] that

$$\overline{C} = \bigcup_{B\sqsubseteq C} B. \tag{3.20}$$

Since we are only interested in one single polytope $\overline{\mathcal{D}}$, we may suppose without loss of generality that *all* the hyperplanes $\mathcal{H}_i$ in the arrangement $\mathcal{A}$ are incident upon $\mathcal{D}$. A polytope is said to be in a *general position* if the corresponding hyperplanes admit the next properties:

$$\text{If} \quad k \leq d, \qquad \text{then} \qquad \dim(\mathcal{H}_1 \cap \cdots \cap \mathcal{H}_k) = d - k, \tag{3.21}$$

$$\text{If} \quad k > d, \qquad \text{then} \qquad \mathcal{H}_1 \cap \cdots \cap \mathcal{H}_k = \emptyset. \tag{3.22}$$

The *Euler-Poincaré characteristic* operator χ is defined to be an integer valued function defined for each V which is a *finite union* of faces as in (3.16) such that χ fulfills the following conditions

(E1) $\chi(\emptyset) = 0$,
(E2) $\chi(F) = (-1)^{\dim(F)}$ for F of the form (3.16),

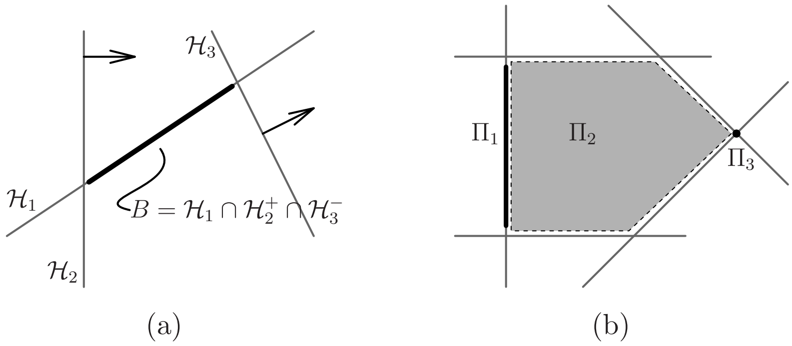


FIGURE 4. (a) The subset $B = \mathcal{H}_1 \cap \mathcal{H}_2^+ \cap \mathcal{H}_3^- \subset \mathbb{R}^2$ has dimension $\dim(B) = \dim(\mathcal{H}_1) = 1$. (b) We have $\Pi_1 \sqsubseteq \Pi_2$ and $\Pi_3 \sqsubseteq \Pi_2$.

(E3) $\chi(V \cup W) = \chi(V) + \chi(W) - \chi(V \cap W)$.

From now on, we will generally adopt the notation of Coxeter and Schläffi by using Π for designing faces of a polytope except the nodes which we denote by N_p.

3.2. **Barycentric coordinates and problem setting.** Consider a polytope $\overline{\mathcal{D}}$ in $\mathbb{R}^d$. According to the *Weyl-Minkowski* property [**?**, **?**], the closure of the polytope $\mathcal{D}$ is the convex hull of the nodes which are incident upon $\mathcal{D}$, i.e.

$$\overline{\mathcal{D}} = \mathrm{HULL}\big\{N \sqsubseteq \mathcal{D} : \quad \dim(N) = 0\big\}. \tag{3.23}$$

We will denote the nodes of the polytope $\mathcal{D}$ by $N_1, ..., N_M$. A barycentric coordinate $\boldsymbol{\lambda}$ is a sequence of functions $(\lambda_1,...,\lambda_M)$

$$\lambda_i : \overline{\mathcal{D}} \longrightarrow \mathbb{R}^+ \cup \{0\} \tag{3.24}$$

such that for each $\mathbf{u} \in \overline{\mathcal{D}}$ we have

$$\sum_{i=1}^{M} \lambda_i(\mathbf{u}) = 1, \qquad \text{and} \qquad \mathbf{u} = \sum_{i=1}^{M} \lambda_i(\mathbf{u}) N_i. \tag{3.25}$$

We also assume the uniqueness property: for given μ_1,..., $\mu_M \in \mathbb{R}^+ \cup \{0\}$ such that $\mu_1 + \cdots + \mu_M = 1$, there is a unique $\mathbf{u} \in \overline{\mathcal{D}}$ such that

$$\lambda_i(\mathbf{u}) = \mu_i. \tag{3.26}$$

As opposed to the simplex case, there might be several methods of defining the functions $\boldsymbol{\lambda}(\mathbf{u})$ which fulfill the two criteria (3.25) and (3.26) in a general convex domain. The purpose of this document is not a review of barycentric coordinates. Joe Warren has done a good survey [**?**] for barycentric coordinates on convex domains. Throughout this document, we *fix one way* of defining barycentric coordinate functions. Because of the uniqueness property (3.26), we will write interchangeably $\mathbf{u} \in \mathcal{D}$ or $\boldsymbol{\lambda} = (\lambda_i) = (\lambda_i(\mathbf{u})) \in \mathcal{D}$ for notational convenience.

Let us now formulate the problem setting accurately. Suppose that we have a boundary function

$$\mathcal{B} : \overline{\mathcal{D}} \setminus \mathcal{D} \longrightarrow \mathbb{R}^p. \tag{3.27}$$

In this document, our intent is to find a transfinite interpolant of $\mathcal{B}$ defined on the whole $\overline{\mathcal{D}}$

$$\mathcal{T}[\mathcal{B}] : \overline{\mathcal{D}} \longrightarrow \mathbb{R}^p \tag{3.28}$$

such that $\mathcal{T}[\mathcal{B}](\boldsymbol{\lambda}) = \mathcal{B}(\boldsymbol{\lambda})$ for all $\boldsymbol{\lambda} \in \overline{\mathcal{D}} \setminus \mathcal{D}$ and that $\mathcal{T}[\mathcal{B}]$ is stable under affine transform such that

$$\mathcal{T}\big[\mathbf{A}(\mathcal{B})\big] = \mathbf{A}\big(\mathcal{T}[\mathcal{B}]\big) \tag{3.29}$$

for any affine transform $\mathbf{A}$.

The affine stability property is very important in applications because the boundary faces are usually given in terms of discrete grids such as control nets of splines or Bézier patches [?, 20]. Suppose that the transfinite interpolation $\mathcal{T}$ of the initial faces has been computed. In order to find the transfinite interpolation with respect to the transformed control points, we need only to apply the affine transform to $\mathcal{T}$.

4. Topologic relations and projections

Before being able to solve the problem in (3.28), we need first to concentrate in this section on the incidence properties of topologic entities. In particular, we will investigate a set $\mathcal{G}(q)$ which is composed of the faces having a node N_q as an incident node. Among others, we will analyze the partitioning of $\mathcal{G}(q)$. Afterwards, a projection will be introduced and we examine some related properties.

4.1. **Incidence information.**

Consider an arrangement $\mathcal{A}$ of hyperplanes $\mathcal{H}_i \subset \mathbb{R}^d$ where $i \in \mathcal{L}$. Let $\overline{\mathcal{D}}$ be a polytope of dimension d which is the closure of the chamber:

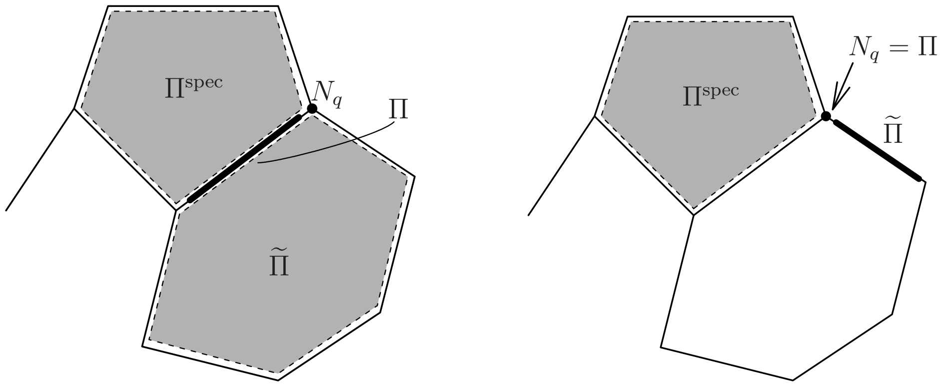
$$\mathcal{D} = \bigcap_{i\in\mathcal{L}} \mathcal{H}_i^{\sigma_i} \qquad \sigma_i \in \{+,-\}. \tag{4.30}$$

For each fixed node N_q of the polytope $\overline{\mathcal{D}}$, we define

$$\mathcal{G}(q) := \big\{\Pi \sqsubseteq \mathcal{D} \quad \text{s.t.} \quad N_q \sqsubseteq \Pi \quad \text{and} \quad \Pi \neq \mathcal{D}\big\}. \tag{4.31}$$

Since the node N_q is a face of $\mathcal{D}$, there must exist $\mathcal{J} = \mathcal{J}(q) \subset \mathcal{L}$ such that

$$N_q = [\cap_{i\in\mathcal{J}(q)}\mathcal{H}_i] \bigcap [\cap_{\mathcal{L}\setminus\mathcal{J}(q)}\mathcal{H}^{\sigma_i}], \tag{4.32}$$



(a) (b)

FIGURE 5. The node N_q is incident upon both Π and $\widetilde{\Pi}$ which are faces of $\mathcal{D}$. We have $\Pi \sqsubseteq \Pi^{\text{spec}}$ while $\widetilde{\Pi} \not\sqsubseteq \Pi^{\text{spec}}$

as introduced in (3.16) where $\dim(\cap_{i\in\mathcal{J}}\mathcal{H}_i) = 0$. Hence, from definition (3.19) we obtain that every face Π of $\mathcal{D}$ such that N_q is incident upon Π must be of the form

$$\Pi \;=\; \Pi_{\mathcal{K}} = \big[\cap_{i\in\mathcal{K}}\mathcal{H}_i\big]\bigcap\big[\cap_{i\in\mathcal{J}\setminus\mathcal{K}}\mathcal{H}_i^{\sigma_i}\big]\bigcap\mathcal{W}_q \in \mathcal{G}(q), \quad \text{where} \tag{4.33}$$

$$\mathcal{W}_q \;:=\; \cap_{\mathcal{L}\setminus\mathcal{J}}\mathcal{H}_i^{\sigma_i} \tag{4.34}$$

in which $\mathcal{K} \subset \mathcal{J}$.

Consider any fixed facet Π^{spec} of the polytope $\overline{\mathcal{D}}$. Since Π^{spec} is of dimension $d-1$, there must exist some $r \in \mathcal{L}$ such that

$$\Pi^{\text{spec}} = \mathcal{H}_r\bigcap\big[\cap_{i\in\mathcal{L}\setminus\{r\}}\mathcal{H}_i^{\sigma_i}\big] \tag{4.35}$$

where $\sigma_i \neq 0$ from (4.30) for all $i \in \mathcal{L}\setminus\{r\}$. Suppose that a node N_q is incident upon Π^{spec}. Let us introduce two sets $\mathcal{G}_1 = \mathcal{G}_1(q,\Pi^{\text{spec}})$ and $\mathcal{G}_2 = \mathcal{G}_2(q,\Pi^{\text{spec}})$ such as

$$\mathcal{G}_1 \;:=\; \Big\{\mathcal{H}_r\bigcap\big[\cap_{i\in\mathcal{P}\setminus\{r\}}\mathcal{H}_i\big]\bigcap\big[\cap_{i\in\mathcal{J}\setminus\mathcal{P}}\mathcal{H}_i^{\sigma_i}\big]\bigcap\mathcal{W}_q : r\in\mathcal{P}\subset\mathcal{J}\Big\} \tag{4.36}$$

$$\mathcal{G}_2 \;:=\; \Big\{\mathcal{H}_r^{\sigma_r}\bigcap\big[\cap_{i\in\mathcal{P}\setminus\{r\}}\mathcal{H}_i\big]\bigcap\big[\cap_{i\in\mathcal{J}\setminus\mathcal{P}}\mathcal{H}_i^{\sigma_i}\big]\bigcap\mathcal{W}_q : r\in\mathcal{P}\subset\mathcal{J}\Big\} \tag{4.37}$$

where the signs σ_i are the same as that in the polytope $\mathcal{D}$ from (4.30). Fig. 5 depicts examples of faces Π and $\widetilde{\Pi}$ which belong respectively to $\mathcal{G}_1$ and $\mathcal{G}_2$.

Before being able to prove the main result of this document, we need to treat several preliminary notions. Let us start from the following easy but important observations.

Lemma 4.1. *Consider a polytope $\overline{\mathcal{D}}$ in general position. For each facet Π^{spec} of the polytope $\mathcal{D}$ and a node N_q incident upon Π^{spec}, the subsets $\mathcal{G}_1(q,\Pi^{\text{spec}})$ and $\mathcal{G}_2(q,\Pi^{\text{spec}})$ from relations (4.36) and (4.37) possess the next properties:*

(C0) $\Pi^{\text{spec}} \in \mathcal{G}_1(q,\Pi^{\text{spec}})$ *and* $\mathcal{D} \in \mathcal{G}_2(q,\Pi^{\text{spec}})$.

(C1) *The sets* $\mathcal{G}_1 \setminus \{\Pi^{\mathrm{spec}}\}$ *and* $\mathcal{G}_2 \setminus \{\mathcal{D}\}$ *form a partition of* $\mathcal{G}(q) \setminus \{\Pi^{\mathrm{spec}}\}$.

(C2) *For each* $\Pi \in \mathcal{G}_1(q, \Pi^{\mathrm{spec}}) \setminus \{\Pi^{\mathrm{spec}}\}$, *there exists a unique* $\widetilde{\Pi} \in \mathcal{G}_2(q, \Pi^{\mathrm{spec}}) \setminus \{\mathcal{D}\}$ *such that*

$$(-1)^{\dim(\Pi)}(-1)^{\dim(\widetilde{\Pi})} = -1. \tag{4.38}$$

PROOF.

The memberships in (C0) are readily obtained from (4.30), (4.35) and the definitions of $\mathcal{G}_1$ and $\mathcal{G}_2$. As for (C1), we observe that every element Π of $\mathcal{G}_1$ can be expressed as $\Pi_{\mathcal{K}}$ in (4.33) by defining $\mathcal{K} := \mathcal{P}$. That holds also for $\mathcal{G}_2$ in which we define $\mathcal{K} := \mathcal{P} \setminus \{r\}$. Thus, we obtain the inclusions

$$\mathcal{G}_1 \setminus \{\Pi^{\mathrm{spec}}\} \subset \mathcal{G}(q) \setminus \{\Pi^{\mathrm{spec}}\}, \qquad \text{and} \qquad \mathcal{G}_2 \setminus \{\mathcal{D}\} \subset \mathcal{G}(q) \setminus \{\Pi^{\mathrm{spec}}\}. \tag{4.39}$$

We want now to show that $\mathcal{G}(q) \subset (\mathcal{G}_1 \setminus \{\Pi^{\mathrm{spec}}\}) \cup (\mathcal{G}_2 \setminus \{\mathcal{D}\})$. Let $\Pi_{\mathcal{K}} \in \mathcal{G}(q)$ be given as in (4.33). We distinguish two cases with respect to $\mathcal{K}$.

Case 1: If $\mathcal{K}$ contains r, then

$$\Pi_{\mathcal{K}} = \mathcal{H}_r \bigcap \left[\, \cap_{i \in \mathcal{K} \setminus \{r\}} \mathcal{H}_i \right] \bigcap \left[\, \cap_{i \in \mathcal{J} \setminus \mathcal{K}} \mathcal{H}_i^{\sigma_i} \right] \bigcap \mathcal{W}_q. \tag{4.40}$$

Hence, $\Pi_{\mathcal{K}}$ belongs to $\mathcal{G}_1$ where $\mathcal{P} := \mathcal{K}$.

Case 2: If $r \notin \mathcal{K}$, then

$$\Pi_{\mathcal{K}} = \mathcal{H}_r^{\sigma_r} \bigcap \left[\, \cap_{i \in \mathcal{K}} \mathcal{H}_i \right] \bigcap \left[\, \cap_{i \in \mathcal{J} \setminus (\mathcal{K} \cup \{r\})} \mathcal{H}_i^{\sigma_i} \right] \bigcap \mathcal{W}_q. \tag{4.41}$$

Hence, $\Pi_{\mathcal{K}}$ belongs to $\mathcal{G}_2$ where $\mathcal{P} := \mathcal{K} \cup \{r\}$.

Since $\mathcal{G}_1$ and $\mathcal{G}_2$ are obviously disjoint, we obtain (C1).

Let $\Pi \in \mathcal{G}_1 \setminus \{\Pi^{\mathrm{spec}}\}$ be given. Thus, there is $\mathcal{P} \subset \mathcal{J}$ such that $r \in \mathcal{P}$ and

$$\Pi = \Pi_{\mathcal{P}} := \mathcal{H}_r \bigcap [\cap_{i \in \mathcal{P} \setminus \{r\}} \mathcal{H}_i] \bigcap [\cap_{i \in \mathcal{J} \setminus \mathcal{P}} \mathcal{H}_i^{\sigma_i}] \bigcap \mathcal{W}_q \in \mathcal{G}_1. \tag{4.42}$$

Let us consider

$$\widetilde{\Pi} = \widetilde{\Pi}_{\mathcal{P}} := \mathcal{H}_r^{\sigma_r} \bigcap [\cap_{i \in \mathcal{P} \setminus \{r\}} \mathcal{H}_i] \bigcap [\cap_{i \in \mathcal{J} \setminus \mathcal{P}} \mathcal{H}_i^{\sigma_i}] \bigcap \mathcal{W}_q \in \mathcal{G}_2. \tag{4.43}$$

Since Π and $\widetilde{\Pi}$ differ only with respect to one single hyperplane $\mathcal{H}_r$, we deduce from the hypothesis of general position (3.21)

$$\dim(\widetilde{\Pi}) = \dim(\Pi) + 1. \tag{4.44}$$

Hence, we obtain (C2).

Q.E.D.

4.2. **Projections onto faces.** For a polytope $\overline{\mathcal{D}}$, let us consider an arbitrary face Π and let C designate the indices of the nodes incident upon Π. Consider now any node N_q where $q \in C$. Let us introduce a projection depending on both Π and N_q:

$$\mathbb{P} := \mathbb{P}_{\Pi,N_q} : \overline{\mathcal{D}} \longrightarrow \overline{\Pi} \tag{4.45}$$

such that for $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_M) \in \overline{\mathcal{D}}$, the barycentric coordinates of the image $\boldsymbol{\mu} := \mathbb{P}(\boldsymbol{\lambda}) = (\mu_1, ..., \mu_M)$ are defined as

$$\mu_j := 0 \quad \text{if} \quad N_j \not\sqsubseteq \Pi \tag{4.46}$$

$$\mu_j := \lambda_j \quad \text{if} \quad N_j \sqsubseteq \Pi \text{ and } j \neq q \quad (\text{i.e.} \quad j \in C \setminus \{q\}) \tag{4.47}$$

$$\mu_q := 1 - \sum_{i \in C \setminus \{q\}} \lambda_i. \tag{4.48}$$

We can immediately observe that $\sum_{j=0}^{d} \mu_j = 1$.

Lemma 4.2. *Consider a face Π^{spec} having dimension $(d-1)$ of a polytope $\overline{\mathcal{D}} \subset \mathbb{R}^d$. Suppose that the nodes incident upon Π^{spec} are $N_{p(1)}$,...,$N_{p(z)}$.*

For every vertex $N_q \sqsubseteq \Pi^{\text{spec}}$, we have

$$\mathbb{P}_{\Pi^{\text{spec}},N_q}(\boldsymbol{\lambda}) = \boldsymbol{\lambda}, \tag{4.49}$$

for each $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_M)$ such that $\lambda_i = 0$ for $N_i \notin \{N_{p(1)}, ..., N_{p(z)}\}$.

PROOF.

Consider $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_M)$ such that $\lambda_i = 0$ for $N_i \notin \{N_{p(1)}, ..., N_{p(z)}\}$ and let us denote $\boldsymbol{\mu} := \mathbb{P}_{\Pi^{\text{spec}},N_q}(\boldsymbol{\lambda})$. We want to show that $\mu_j = \lambda_j$ in which we examine several situations about $j = 1, ..., M$. First, consider $j \notin \{p(1), ..., p(z)\}$. Thus, we have $\lambda_j = 0$. Since $N_j \not\sqsubseteq \Pi^{\text{spec}}$, we deduce from (4.46) that $\mu_j = 0$. Hence, $\mu_j = \lambda_j$. Consider now an index $j \in \{p(1), ..., p(z)\}$ such that $N_j \neq N_q$. By the definition of $\mathbb{P}$ in (4.47), we obtain $\mu_j = \lambda_j$.

On the other hand, due to the partition of unity, we obtain:

$$\mu_q = 1 - \sum_{i \in \{p(1),...,p(z)\} \setminus \{q\}} \mu_i = 1 - \sum_{i \in \{p(1),...,p(z)\} \setminus \{q\}} \lambda_i \tag{4.50}$$

$$= 1 - \sum_{\substack{i=1 \\ i \neq q}}^{N} \lambda_i = \lambda_q. \tag{4.51}$$

That means, for any $j = 1, ..., M$ we always have $\mu_j = \lambda_j$. As a consequence, we obtain $\mathbb{P}_{\Pi^{\text{spec}},N_q}(\boldsymbol{\lambda}) = \boldsymbol{\lambda}$ for such $\boldsymbol{\lambda}$.

Q.E.D.

Lemma 4.3. *Consider a face* $\Pi^{\rm spec}$ *having dimension* $(d-1)$ *of a polytope* $\overline{\mathcal{D}} \subset \mathbb{R}^d$ *which is in general position. Let a node* N_q *be incident upon* $\Pi^{\rm spec}$*. Suppose that* $\Pi^{\rm spec}$ *is supported by the hyperplane* $\mathcal{H}_r$*. Consider two faces*

$$\Pi = \Pi_{\mathcal{P}} := \mathcal{H}_r \bigcap [\cap_{i\in\mathcal{P}\setminus\{r\}}\mathcal{H}_i] \bigcap [\cap_{i\in\mathcal{J}\setminus\mathcal{P}}\mathcal{H}_i^{\sigma_i}] \bigcap \mathcal{W}_q, \tag{4.52}$$

$$\widetilde{\Pi} = \widetilde{\Pi}_{\mathcal{P}} := \mathcal{H}_r^{\sigma_r} \bigcap [\cap_{i\in\mathcal{P}\setminus\{r\}}\mathcal{H}_i] \bigcap [\cap_{i\in\mathcal{J}\setminus\mathcal{P}}\mathcal{H}_i^{\sigma_i}] \bigcap \mathcal{W}_q \tag{4.53}$$

such that $\dim(\Pi)$ *and* $\dim(\widetilde{\Pi})$ *have different parity. We have the following identity*

$$(-1)^{\dim(\Pi)}\mathcal{B}\circ\mathbb{P}_{\Pi,N_q}(\boldsymbol{\lambda}) + (-1)^{\dim(\widetilde{\Pi})}\mathcal{B}\circ\mathbb{P}_{\widetilde{\Pi},N_q}(\boldsymbol{\lambda}) = 0 \tag{4.54}$$

for all $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_M)$ *such that* $\lambda_i = 0$ *for* $N_i \not\sqsubseteq \Pi^{\rm spec}$.

PROOF.

Observe first that $\Pi \sqsubseteq \Pi^{\rm spec}$ whereas $\widetilde{\Pi} \not\sqsubseteq \Pi^{\rm spec}$. Consider $\boldsymbol{\mu} := \mathbb{P}_{\Pi,N_q}(\boldsymbol{\lambda})$ and $\widetilde{\boldsymbol{\mu}} := \mathbb{P}_{\widetilde{\Pi},N_q}(\boldsymbol{\lambda})$. We intend to demonstrate that $\mu_j = \widetilde{\mu}_j$ for each $j = 1, ..., M$. We will consider several cases with respect to $j = 1, ..., M$.

Case 1: If $N_j \not\sqsubseteq \Pi^{\rm spec}$ then by hypothesis we have $\lambda_j = 0$. Since $\Pi \sqsubseteq \Pi^{\rm spec}$, N_j is not incident upon Π either. Hence, according to (4.46) about the definition of $\mathbb{P}$ we obtain $\mu_j = 0$. As for the determination of $\widetilde{\mu}_j$, we consider two situations. If $N_j \sqsubseteq \widetilde{\Pi}$, we obtain from (4.47) that $\widetilde{\mu}_j = \lambda_j = 0$. In the situation $N_j \not\sqsubseteq \widetilde{\Pi}$, we deduce from (4.46) that $\widetilde{\mu}_j = 0$. Thus, in both situations, we have $\widetilde{\mu}_j = \mu_j$.

Case 2: Suppose now that $N_j \sqsubseteq \Pi^{\rm spec}$ and that $N_j \neq N_q$. We consider again two situations. In the situation where $N_j \not\sqsubseteq \widetilde{\Pi}$, we have $\widetilde{\mu}_j = 0$. Since $\Pi \sqsubseteq \widetilde{\Pi}$, we have also $N_j \not\sqsubseteq \Pi$. Thus, $\mu_j = 0$. Hence, $\widetilde{\mu}_j = \mu_j$. Consider now the situation where $N_j \sqsubseteq \widetilde{\Pi}$. That is $\widetilde{\mu}_j = \lambda_j$. Since N_j is a face of $\mathcal{D}$, we have from (3.16)

$$N_j = \cap_{i\in\mathcal{L}}\mathcal{H}_i^{z_i} \tag{4.55}$$

where $z_i \in \{0, \sigma_i\}$. Since $N_j \sqsubseteq \Pi^{\rm spec}$ which is supported by $\mathcal{H}_r$, $z_r = 0$ so that relation (4.55) becomes

$$N_j = \mathcal{H}_r \bigcap [\cap_{i\in\mathcal{L}\setminus\{r\}}\mathcal{H}_i^{z_i}]. \tag{4.56}$$

We apply the same process to $N_j \sqsubseteq \widetilde{\Pi}$ in order to obtain

$$N_j = \mathcal{H}_r \bigcap [\cap_{i\in\mathcal{P}\setminus\{r\}}\mathcal{H}_i] \bigcap [\cap_{i\in\mathcal{L}\setminus\mathcal{P}}\mathcal{H}_i^{z_i}] \tag{4.57}$$

where $z_i \in \{0, \sigma_i\}$ if $i \in \mathcal{L}\setminus\mathcal{P}$. That is to say $N_j \sqsubseteq \Pi$. As a consequence, $\mu_j = \lambda_j$ or equivalently $\mu_j = \widetilde{\mu}_j$.

Case 3: Suppose now that $N_j = N_q$. From the first two cases, we immediately obtain

$$\widetilde{\mu}_q = 1 - \sum_{\substack{i=1\\ i\neq q}}^{M} \widetilde{\mu}_i = 1 - \sum_{\substack{i=1\\ i\neq q}}^{M} \mu_i = \mu_q. \tag{4.58}$$

Therefore, we obtain $\mathbb{P}_{\Pi,N_q}(\boldsymbol{\lambda}) = \mathbb{P}_{\widetilde{\Pi},N_q}(\boldsymbol{\lambda})$. As a result, we deduce

$$(-1)^{\dim(\Pi)}\mathcal{B}\circ\mathbb{P}_{\Pi,N_q}(\boldsymbol{\lambda}) = (-1)^{\dim(\Pi)}\mathcal{B}\circ\mathbb{P}_{\widetilde{\Pi},N_q}(\boldsymbol{\lambda}). \tag{4.59}$$

Q.E.D.

5. Construction of the transfinite interpolation

In the previous sections, we saw different definitions and lemmas for the generation of the projections on faces. Now, we want to use them to construct a transfinite interpolation on the whole polytope $\overline{\mathcal{D}}$.

5.1. **Blending functions and topologic formula.** To generalize the case of 2D Coons map, let us introduce the notion of *barycentric blending functions* defined for all $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_M)$ in the reference domain $\overline{\mathcal{D}}$. In fact, we define them to be a set of functions b_i, $i = 1, ..., M$ admitting the following three properties:

$$(P1) \quad \text{If} \quad \lambda_p = 0 \quad \text{in} \quad \boldsymbol{\lambda} = (\lambda_1, ..., \lambda_M) \quad \text{then} \quad b_p(\boldsymbol{\lambda}) = 0, \tag{5.60}$$

$$(P2) \quad \text{For each } p = 1, ..., M, \text{ we have} \quad b_p(\boldsymbol{\lambda}) = 1 \quad \text{if} \quad \lambda_j = \delta_{j,p}, \tag{5.61}$$

$$(P3) \quad \sum_{i=1}^{M} b_i(\boldsymbol{\lambda}) = 1 \qquad \forall\, \boldsymbol{\lambda} = (\lambda_1, ..., \lambda_M) \in \mathcal{D}. \tag{5.62}$$

As in the case of 2D transfinite interpolations, the simplest way of choosing the blending functions is to use the linear ones which are in the current case:

$$b_i(\boldsymbol{\lambda}) := \lambda_i, \qquad \forall\, i = 1, ..., M. \tag{5.63}$$

Theorem 5.1. *Consider a polytope $\overline{\mathcal{D}} \subset \mathbb{R}^d$ in general position having M nodes N_1,..., N_M. The function*

$$\mathcal{T}[\mathcal{B}](\boldsymbol{\lambda}) = (-1)^{d+1}\sum_{i=1}^{M} b_i(\boldsymbol{\lambda}) \sum_{\Pi\in\mathcal{G}(i)} (-1)^{\dim(\Pi)}\mathcal{B}\circ\mathbb{P}_{\Pi,N_i}(\boldsymbol{\lambda}) \tag{5.64}$$

determines a transfinite interpolation. Thus, for a facet Π^{spec} of $\mathcal{D}$ such that the nodes of Π^{spec} are $N_{p(1)}$, ...,$N_{p(z)}$, we have

$$\mathcal{T}[\mathcal{B}](\boldsymbol{\lambda}) = \mathcal{B}(\boldsymbol{\lambda}) \tag{5.65}$$

for every $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_M)$ such that $\lambda_q = 0$ for all $q \notin \{p(1), ..., p(z)\}$.

PROOF.

Let us consider $\boldsymbol{\lambda}$ such that $\lambda_q = 0$ for all $q \notin V := \{p(1), ..., p(z)\}$. We obtain from property (P1) of the barycentric blending functions b_i in (5.60) that

$$\mathcal{T}[\mathcal{B}](\boldsymbol{\lambda}) = (-1)^{d+1} \sum_{i=1}^{M} b_i(\boldsymbol{\lambda}) \sum_{\Pi \in \mathcal{G}(i)} (-1)^{\dim(\Pi)} \mathcal{B} \circ \mathbb{P}_{\Pi, N_i}(\boldsymbol{\lambda}) \tag{5.66}$$

$$= (-1)^{d+1} \sum_{q \in V} b_q(\boldsymbol{\lambda}) \sum_{\Pi \in \mathcal{G}(q)} (-1)^{\dim(\Pi)} \mathcal{B} \circ \mathbb{P}_{\Pi, N_q}(\boldsymbol{\lambda}). \tag{5.67}$$

Let us denote by $\mathcal{F}(q, \Pi^{\text{spec}})$ the set $\mathcal{G}(q) \setminus \{\Pi^{\text{spec}}\}$ so that we obtain

$$\mathcal{T}[\mathcal{B}](\boldsymbol{\lambda}) = (-1)^{d+1} \sum_{q \in V} b_q(\boldsymbol{\lambda}) \Big[(-1)^{d-1} \mathcal{B} \circ \mathbb{P}_{\Pi^{\text{spec}}, N_q(\boldsymbol{\lambda})} + \sum_{\Pi \in \mathcal{F}(q, \Pi^{\text{spec}})} (-1)^{\dim(\Pi)} \mathcal{B} \circ \mathbb{P}_{\Pi, N_q}(\boldsymbol{\lambda}) \Big].$$

We use now Lemma 4.1 to partition $\mathcal{F}(q, \Pi^{\text{spec}})$ into $\mathcal{F}_1(q, \Pi^{\text{spec}}) := \mathcal{G}_1(q, \Pi^{\text{spec}}) \setminus \{\Pi^{\text{spec}}\}$ and $\mathcal{F}_2(q, \Pi^{\text{spec}}) := \mathcal{G}_2(q, \Pi^{\text{spec}}) \setminus \{\mathcal{D}\}$. Consequently, we break up the above summation as

$$\mathcal{T}[\mathcal{B}](\boldsymbol{\lambda}) = (-1)^{d+1} \sum_{q \in V} b_q(\boldsymbol{\lambda}) \Big[(-1)^{d-1} \mathcal{B} \circ \mathbb{P}_{\Pi^{\text{spec}}, N_q(\boldsymbol{\lambda})} + \sum_{\Pi \in \mathcal{F}_1(q, \Pi^{\text{spec}})} (-1)^{\dim(\Pi)} \mathcal{B} \circ \mathbb{P}_{\Pi, N_q}(\boldsymbol{\lambda}) + \sum_{\Pi \in \mathcal{F}_2(q, \Pi^{\text{spec}})} (-1)^{\dim(\Pi)} \mathcal{B} \circ \mathbb{P}_{\Pi, N_q}(\boldsymbol{\lambda}) \Big].$$

As a result, we obtain

$$\mathcal{T}(\boldsymbol{\lambda}) = (-1)^{d+1} \sum_{q \in V} b_q(\boldsymbol{\lambda}) \Big[(-1)^{d-1} \mathcal{B} \circ \mathbb{P}_{\Pi^{\text{spec}}, N_q}(\boldsymbol{\lambda}) + \sum_{\Pi \in \mathcal{F}_1(q, \Pi^{\text{spec}})} \Big((-1)^{\dim(\Pi)} \mathcal{B} \circ \mathbb{P}_{\Pi, N_q}(\boldsymbol{\lambda}) + (-1)^{\dim(\widetilde{\Pi})} \mathcal{B} \circ \mathbb{P}_{\widetilde{\Pi}, N_q}(\boldsymbol{\lambda}) \Big) \Big].$$

where $\widetilde{\Pi} \in \mathcal{F}_2(q, \Pi^{\text{spec}})$ is the unique face corresponding to $\Pi \in \mathcal{F}_1(q, \Pi^{\text{spec}})$ as specified in (4.38). Due to a combination of property (C2) in Lemma 4.1 and Lemma 4.3, the last two terms cancel out simultaneously, so that we obtain

$$\mathcal{T}(\boldsymbol{\lambda}) = (-1)^{d+1} \sum_{q \in V} (-1)^{d-1} b_q(\boldsymbol{\lambda}) \, \mathcal{B} \circ \mathbb{P}_{\Pi^{\text{spec}}, N_q}(\boldsymbol{\lambda}). \tag{5.68}$$

According to relation (4.49) of Lemma 4.2, $\mathbb{P}_{\Pi^{\text{spec}}, N_q}(\boldsymbol{\lambda}) = \boldsymbol{\lambda}$ irrespective of $q \in V$. Hence,

$$\mathcal{T}(\boldsymbol{\lambda}) = \mathcal{B}(\boldsymbol{\lambda}) \sum_{q \in V} b_q(\boldsymbol{\lambda}) = \mathcal{B}(\boldsymbol{\lambda}). \tag{5.69}$$

Q.E.D.

5.2. **Affine stability.** In this section, we would like to concentrate on the proof of the affine stability of the proposed transfinite interpolation.

Theorem 5.2. *For the d-dimensional polytope $\mathcal{D}$ in general position having nodes $N_1,\ldots, N_M$, the interpolant*

$$\mathcal{T}[\mathcal{B}](\boldsymbol{\lambda}) := (-1)^{d+1} \sum_{q=1}^{M} b_q(\boldsymbol{\lambda}) \sum_{\Pi\in\mathcal{G}(q)} (-1)^{\dim(\Pi)} \mathcal{B}\circ \mathbb{P}_{\Pi,N_q}(\boldsymbol{\lambda}) \tag{5.70}$$

is affinely stable. Thus, we have

$$\mathcal{T}\big[\mathbf{A}(\mathcal{B})\big] = \mathbf{A}\big(\mathcal{T}[\mathcal{B}]\big) \tag{5.71}$$

for any affine transform $\mathbf{A}$.

PROOF.

Let us consider an affine transform $\mathbf{A} : \mathbb{R}^p \longrightarrow \mathbb{R}^{\tilde{p}}$. Thus, we have the property that $\mathbf{A}\Big(\sum_{j\in I} \mu_j v_j\Big) = \sum_{j\in I} \mu_j \mathbf{A}(v_j)$ for any coefficients $(\mu_j)_{j\in I}$ such that $\sum_{j\in I} \mu_j = 1$.

For each node N_q incident upon the polytope $\mathcal{D}$, let us again consider $\mathcal{J}(q) \subset \mathcal{L}$ such that $N_q = [\cap_{i\in\mathcal{J}(q)}\mathcal{H}_i] \bigcap [\cap_{\mathcal{L}\setminus\mathcal{J}(q)}\mathcal{H}^{\sigma_i}]$ as introduced in relation (4.32). Hence, every face Π of $\mathcal{D}$ upon which N_q is incident must be of the form

$$\Pi = \Pi_{\mathcal{K}} := \big[\cap_{i\in\mathcal{K}} \mathcal{H}_i\big] \bigcap \big[\cap_{i\in\mathcal{J}\setminus\mathcal{K}} \mathcal{H}_i^{\sigma_i}\big] \bigcap \mathcal{W}_q \tag{5.72}$$

where $\mathcal{K} \subset \mathcal{J}$. On the one hand, by applying (3.20) to the subarrangement

$$\mathcal{B} := \{\mathcal{H}_i \quad \text{where} \quad i\in\mathcal{J}\} \subset \mathcal{A}, \tag{5.73}$$

we obtain the topologic closure

$$\overline{\bigcap_{i\in\mathcal{J}} \mathcal{H}_i^{\sigma_i}} = \bigcup_{\mathcal{K}\subset\mathcal{J}} \Big(\big[\cap_{i\in\mathcal{K}} \mathcal{H}_i\big] \bigcap \big[\cap_{i\in\mathcal{J}\setminus\mathcal{K}} \mathcal{H}_i^{\sigma_i}\big]\Big). \tag{5.74}$$

(Observe that (5.74) is not the union over $\mathcal{K} \subset \mathcal{J}$ of $\Pi_{\mathcal{K}}$ because we have $\mathcal{L}\setminus\mathcal{K}$ in (5.72) an intersection with $\mathcal{W}_q$).

As a consequence, since we took the union of disjoint sets in (5.74) we obtain

$$\begin{aligned}
\chi\Big(\overline{\bigcap_{i\in\mathcal{J}} \mathcal{H}_i^{\sigma_i}}\Big) &= \sum_{\mathcal{K}\subset\mathcal{J}} \chi\Big(\big[\cap_{i\in\mathcal{K}} \mathcal{H}_i\big] \bigcap [\cap_{i\in\mathcal{J}\setminus\mathcal{K}}\mathcal{H}_i^{\sigma_i}]\Big) \\
&= \sum_{\mathcal{K}\subset\mathcal{J}} \exp_{-1}\dim\Big(\big[\cap_{i\in\mathcal{K}} \mathcal{H}_i\big] \bigcap [\cap_{i\in\mathcal{J}\setminus\mathcal{K}}\mathcal{H}_i^{\sigma_i}]\Big) \\
&= \sum_{\mathcal{K}\subset\mathcal{J}} \exp_{-1}\dim\Big(\big[\cap_{i\in\mathcal{K}} \mathcal{H}_i\big] \bigcap [\cap_{i\in\mathcal{L}\setminus\mathcal{K}}\mathcal{H}_i^{\sigma_i}] \bigcap \mathcal{W}_q\Big) \\
&= \sum_{\mathcal{K}\subset\mathcal{J}} (-1)^{\dim\big(\Pi_{\mathcal{K}}\big)}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}\chi\Big(\overline{\bigcap_{i\in\mathcal{J}}\mathcal{H}_i^{\sigma_i}}\Big) &= \chi(\mathbb{R}^d)-\chi\big(\mathbb{R}^d\setminus\overline{\bigcap_{i\in\mathcal{J}}\mathcal{H}_i^{\sigma_i}}\big)\\ &= (-1)^d-\chi\big(\bigcup_{i\in\mathcal{J}}\mathcal{H}_i^{\rho_i}\big)\end{aligned}$$

where $\rho_i := -\sigma_i$. From the inclusion-exclusion property of the Euler-Poincaré operator, we have

$$\begin{aligned}\chi(\bigcup_{i\in\mathcal{J}}\mathcal{H}_i^{\rho_i}) &= \sum_{i\in\mathcal{J}}\chi(\mathcal{H}_i^{\rho_i})-\sum_{\substack{i<j\\ i,j\in\mathcal{J}}}\chi(\mathcal{H}_i^{\rho_i}\cap\mathcal{H}_j^{\rho_j})+\\ &+\sum_{\substack{i<j<k\\ i,j,k\in\mathcal{J}}}\chi(\mathcal{H}_i^{\rho_i}\cap\mathcal{H}_j^{\rho_j}\cap\mathcal{H}_k^{\rho_k})-\cdots+(-1)^{\mathrm{card}(\mathcal{J})}\chi(\bigcap_{i\in\mathcal{J}}\mathcal{H}_i^{\rho_i}).\end{aligned}$$

For any subset $\mathcal{R}\subset\mathcal{J}$, we have $\bigcap_{i\in\mathcal{R}}\mathcal{H}_i^{\rho_i}\neq\emptyset$ because each $\mathcal{H}_i$ contains N_q for $i\in\mathcal{J}$. Therefore, we have $\dim(\bigcap_{i\in\mathcal{R}}\mathcal{H}_i^{\rho_i})=d$ because $\rho_i\neq 0$. Hence, $\chi(\bigcap_{i\in\mathcal{R}}\mathcal{H}_i^{\rho_i})=(-1)^d$. As a consequence, we obtain

$$\chi(\bigcup_{i\in\mathcal{J}}\mathcal{H}_i^{\rho_i}) = \sum_{j=1}^{\mathrm{card}(\mathcal{J})}(-1)^{j-1}(-1)^d\binom{\mathrm{card}(\mathcal{J})}{j} \tag{5.75}$$

$$= (-1)^{d+1}\sum_{j=1}^{\mathrm{card}(\mathcal{J})}(-1)^j\binom{\mathrm{card}(\mathcal{J})}{j} \tag{5.76}$$

$$= (-1)^{d+1}(-1)=(-1)^d. \tag{5.77}$$

Hence, $\chi(\bigcup_{i\in\mathcal{J}}\mathcal{H}_i^{\rho_i})=(-1)^d$. That implies $\chi\Big(\overline{\bigcap_{i\in\mathcal{J}}\mathcal{H}_i^{\sigma_i}}\Big)=0$. Hence, we obtain $\sum_{\mathcal{K}\subset\mathcal{J}}(-1)^{\dim\left(\Pi_{\mathcal{K}}\right)}=0$. That is,

$$(-1)^{\dim(\mathcal{D})}+\sum_{\Pi\in\mathcal{G}(q)}(-1)^{\dim(\Pi)}=0 \tag{5.78}$$

which means

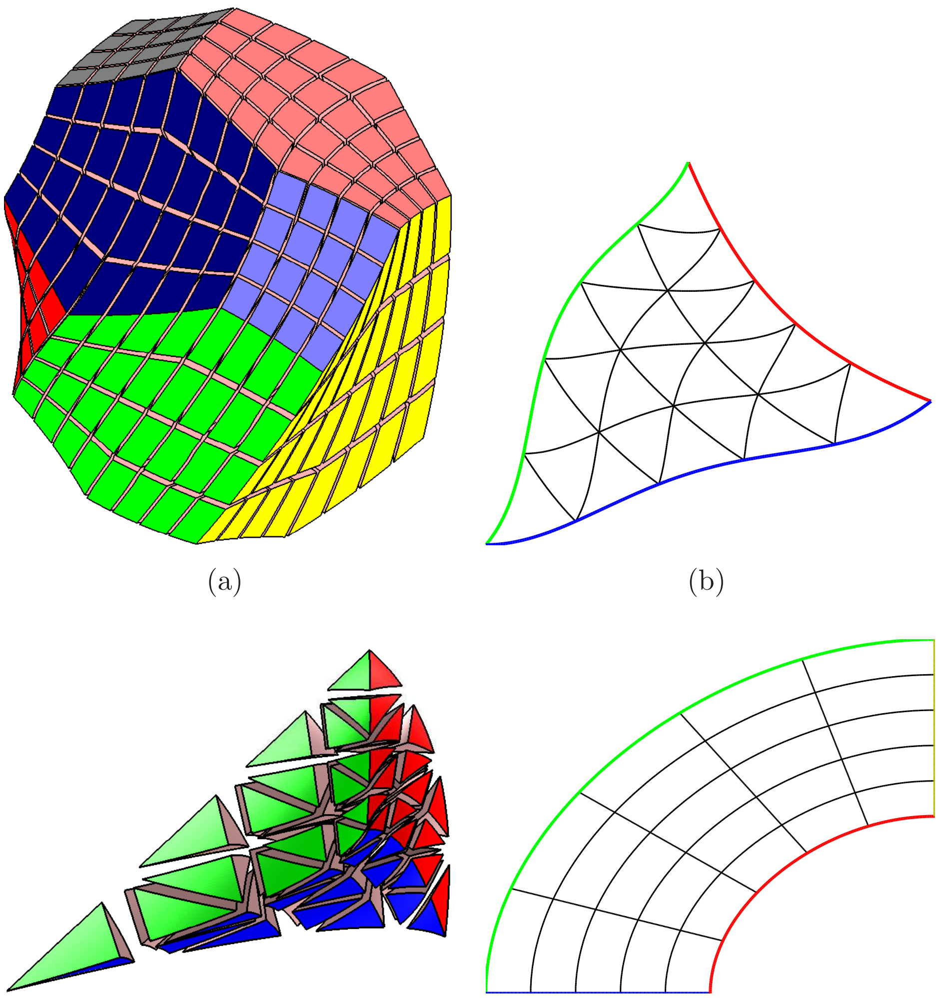
$$(-1)^{d+1}\sum_{\Pi\in\mathcal{G}(q)}(-1)^{\dim(\Pi)}=1. \tag{5.79}$$

As a consequence, we obtain that the coefficients sum to unity such as

$$(-1)^{d+1}\sum_{q=1}^{M}b_q(\boldsymbol{\lambda})\sum_{\Pi\in\mathcal{G}(q)}(-1)^{\dim(\Pi)}=\sum_{q=1}^{M}b_q(\boldsymbol{\lambda})=1. \tag{5.80}$$

which implies the affine stability (5.71).

Q.E.D.



(a) (b)

(c) (d)

FIGURE 6. Transfinite interpolation defined on some convex polytopes. In (d) we recognize the usual Coons patch where the polygon of definition is the unit square.

5.3. **Conclusion and illustrative results.** In order to solve the problem of transfinite interpolation, we have proposed a topologic formula of the form

$$(-1)^{d+1}\sum_{i=1}^{M} b_i(\boldsymbol{\lambda}) \sum_{\Pi\in\mathcal{G}(i)} (-1)^{\dim(\Pi)}\mathcal{B}\circ\mathbb{P}_{\Pi,N_i}(\boldsymbol{\lambda}) \tag{5.81}$$

where $\mathcal{G}(i)$ is the set of topologic entities upon which the node N_i is incident. In contrast to some other methods, the presented technique enables the treatment of multifaceted domains which are not necessarily of tensor product structure. Doing further analytical works with such short formula should be more practical than using long ones. Therefore, we believe that the above method has good applications in theoretical analysis such as search for upper bounds or error estimations. Unfortunately, we could only prove the result under the condition of general position of the polytopes. We think that such a condition is only redundant and the formula remains valid without any positional restriction but the proof for that seems to be very difficult. That will be an open question which we might treat in the future. On the other hand, the above method can also be used in applied situations. For instance, let us briefly show some practical results from the formerly proposed approach. Note that the results shown here are not meant to be a substitution of the demonstrations. They are only for illustrative purpose. We have considered some transfinite interpolations where the boundary curves or surfaces are given. We applied the above transfinite interpolation to find the images inside the convex domains. In Fig.6, we have gathered the results which consist of the image by the transfinite interpolation of some discretization within some convex domains.

Acknowledgment

During the PhD work of the author, Prof. Brunnett (University of Chemnitz) has helped him in the topic of computational geometry. Prof. Schneider (University of Berlin) and Prof. Harbrecht (University of Bonn) have supported the author in his Post-doctoral positions. The author is grateful to the HCM (Hausdorff Center for Mathematics) and its Excellence Cluster program for supporting his visit at the University of Bonn.

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